	COOMET Recommendation	COOMET R/GM/35:2022
	Expression of the expanded measurement uncertainty (method of kurtosis)	

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1. INTRODUCTION

In 1993, the "Guide to the Expression of Uncertainty of Measurement" (GUM) [1] was published, which was based on: the law of propagation of uncertainty, leading to a bias in estimates of the numerical values of the measurand and its uncertainty under nonlinear model equations; central limit theorem of the theory of probability with the apparatus of the number of degrees of freedom, predetermining the unreliability of estimates of expanded uncertainty due to ignoring the influence of the distribution laws of input quantities on the distribution law of the measurand.

The introduction of Appendices 1 [2] and 2 [3] to the GUM, based on the Monte Carlo method (MCM), made it possible to get rid of these shortcomings. The estimates of measurement uncertainty obtained using the MCM correspond to Bayesian estimates, but differ numerically from the estimates obtained using the GUM approach. It should be noted that the following factors impede the direct use of MCM for assessing measurement uncertainty in testing and calibration laboratories accredited for compliance with the requirements of ISO/IEC 17025:2017 [4]:

- lack of specialized certified software for estimating measurement uncertainty based on MCM;
- impossibility of obtaining the measurement uncertainty budget by the existing software implementing MCM;
- impossibility of documenting a step-by-step procedure for estimating measurement uncertainty based on MCM.

This document describes the procedure for estimating measurement uncertainty, based on the method of kurtosis, the method of finite increments and the law of propagation of expanded uncertainty, which make it possible to obtain estimates of the numerical values and the uncertainty of the measurement result comparable to the estimates obtained using the MCM.

2. CONDITIONS OF USE

The procedures in this recommendation are applicable to the measurement uncertainty evaluation when the following conditions are met:

1. Measurement model is real, explicit, one-dimensional, linear or linearizable.
2. The probability density function (PDF) of input quantities is symmetric.
3. The PDF of the total population of the observed dispersion of measuring instruments (MI) readings is Gaussian.

3. NOTATION USED

- c_i is sensitivity coefficient;
- c_{ii} is partial derivative of the second order of Y with respect to X_i, which is estimated at $X_1 = x_1, ..., X_N = x_N$;
- c_{ij} is mixed second order partial derivative of Y with respect to X_i, X_j, which is estimated at $X_1 = x_1, ..., X_N = x_N$;
- $\mathrm{cov}(x_i, x_j)$ is covariance of the measurement results of the *i*-th and *j*-th input quantities;
- k_p is coverage factor for confidence level p;
- n_i is number of repeated measurements of the *i*-th input quantity;
- N is number of input quantities in the measurement model ;
- p is confidence level;
- r_{kl} is correlation coefficient between the results of measurement X_k, X_l;
- s_i is standard deviation of the *i*-th input quantity;
- $s(\overline{x}_i)$ is standard deviation of the arithmetic mean of the *i*-th input quantity;
- $u(x_i) = u_i$ is standard uncertainty of the *i*-th input quantity;
- $u(y)$ is standard uncertainty of a measurand;
- U is expanded uncertainty of a measurand;
- U_A is type A expanded uncertainty of a measurand;
- U_B is type B expanded uncertainty of a measurand;
- X_i is the *i*-th input quantity of the measurement model;
- x_i is numerical value of the *i*-th input quantity;
- x_{ir} is *r*-th repeated reading when measuring the *i*-th input quantity;
- Y is measurand (output quantity) of the measurement model;
- y is numerical value of the measurand;
- α is parameter of the trapezoidal PDF;
- β is relative deviation of inaccurately specified boundaries of uniform PDF;
- η_i is kurtosis of the *i*-th input quantity;
- η is kurtosis of the measurand;
- γ_i is coefficient for converting the boundaries of the distribution of the *i*-th input quantity into the standard uncertainty;
- δ_i is corrections for the *i*-th influencing quantity;
- $\Delta(y)$ is bias of the measurand;
- $\Delta(u^2)$ is bias of the measurand variance;
- ε_i is correction for the *i*-th random measurement error;
- θ_i is boundary of the *i*-th input quantity PDF;
- ν is number of degrees of freedom of the Student’s PDF.

4. GENERAL PROVISIONS

4.1. Measurement model

A real, explicit, one-dimensional measurement model is written as:

$$Y = f(X_1, X_2, ..., X_N), \tag{1}$$

where Y – measurand; $X_1, X_2, ..., X_N$ – input quantities.

The input quantities can be classified as:

- quantities, values and uncertainties of which are determined directly in this measurement and can be obtained from a single indication of measuring instruments (MI) or repeated indications;
- corrections for MI readings and corrections for influencing input quantities such as ambient temperature, barometric pressure, humidity and others;
- quantities, whose values and uncertainties are entered into measurements from external sources, such as calibrated standards, certified reference materials and data specified in reference books .

4.2. Evaluation of input quantities, their standard uncertainties and covariances

If a single MI reading x_i of the input quantity X_i is obtained directly in this measurement, then this reading is the value of this quantity.

The instrumental standard uncertainty of this quantity is obtained from information taken from the MI calibration certificate – expanded instrumental uncertainty U_{pi} and coverage factor k_{pi}:

$$u(x_i) = \frac{U_{pi}}{k_{pi}}, \tag{2}$$

where p is the confidence level, which is usually "about 0,95", that is, exactly 0,9545. For this probability, information about the probability density distribution (PDF) that is assigned to this input quantity can be obtained from Table 1.

Table 1

Coverage factor for confidence level $p = 0,9545$ and corresponding PDF*

	1,411	1,653	1,653…1,927	1,927	2	>2
DF	Arcsine	Uniform	Trapezoidal	Triangular	Gaussian	Student's

**NOTE. Nomograms for finding the parameter α of the trapezoidal PDF and the effective number of degrees of freedom ν for the Student's PDF for a probability of 0,9545 are given in Appendix A.*

If there is a priori information about the variability of a single MI reading x_i of an input quantity X_i, characterized by a standard deviation s_i, it is necessary to add a correction ε_i to the measurement model for a random error, the values of which $\hat{\varepsilon}_i = 0$, and the standard uncertainty $u(\varepsilon_i)$ is equal to $s_i\sqrt{\nu_i/(\nu_i - 2)}$, where ν_i is the number of degrees of freedom attributed to the *i*-th correction.

If n_i of repeating SI readings $x_{i1}, x_{i2}, ..., x_{in}$ are obtained directly in this measurement, then the assessment of this value X_i is their arithmetic mean:

$$x_i = \bar{x}_i = \frac{1}{n_i}\sum_{r=1}^{n_i} x_{ir} \,. \tag{3}$$

In this case, a random error correction ε_i is added to the measurement model, the value of which $\hat{\varepsilon}_i = 0$, and the standard uncertainty, is determined by the formula:

$$u(\varepsilon_i) = \sqrt{\frac{1}{n_i(n_i-1)}\sum_{r=1}^{n_i}(x_{i,r}-\bar{x}_i)^2}\sqrt{\frac{n_i-1}{n_i-3}} = \sqrt{\frac{1}{n_i(n_i-3)}\sum_{r=1}^{n_i}(x_{i,r}-\bar{x}_i)^2} \,. \tag{4}$$

An unbiased scaled Student's PDF with the number of degrees of freedom $\nu_i = n_i - 1$ is attributed to this amendment in [5]. Estimates of standard uncertainties (4) of corrections ε_i make sense at $n_i \geq 4$ and are valid only for normally distributed results of multiple measurements [5].

If there is a correlation between the measurements of two input quantities X_k, X_l, then the estimate of their covariance is found by the formula:

$$\mathrm{cov}(x_k, x_l) = r_{kl} u(\varepsilon_k) u(\varepsilon_l) \,, \tag{5}$$

where r_{kl} is the estimate of the correlation coefficient, made according to the formula:

$$r_{kl} = \frac{\sum_{r=1}^{n}(x_{kr}-\bar{x}_k)(x_{lr}-\bar{x}_l)}{\sqrt{\sum_{r=1}^{n}(x_{kr}-\bar{x}_k)^2\sum_{r=1}^{n}(x_{lr}-\bar{x}_l)^2}} \,. \tag{6}$$

Corrections for influencing input quantities δ_i having an estimated value $\hat{\delta}_i$ and standard uncertainty u_i are introduced by additive or multiplicative correction of the corresponding input or measured quantity.

If the input value X_i is specified by the boundaries $\pm\theta_i$ of its variability, then its standard uncertainty is found by the formula:

$$u_i = \frac{\theta_i}{\gamma_i} \,, \tag{7}$$

Where γ_i is the coefficient determined by PDF of X_i within the boundaries of its variability (Table 2).

Table 2

Coefficients γ for converting the boundaries $\pm\theta_i$ of the input quantity PDF into the standard uncertainty

PDF	Arcsine	Uniform	Triangular
γ	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{6}$

Values x_i and standard uncertainties u_i based on a priori information obtained from external sources are assigned to quantities X_i that are entered into measurements from these sources, such as calibrated standards, certified reference materials and data specified in reference books,.

PDFs are a priori attributed to all input values, which are characterized by kurtosis η_i, the values of which are given in Table 3.

The rationale for the choice of the distribution law of the input quantity based on the available information is given in [2].

Table 3

Kurtosis of the PDF of input quantities

PDF	Value of kurtosis
Arcsine	$-1,5$
Uniform	$-1,2$
Uniform with inaccurately defined boundaries (β – relative deviation of the boundaries of a uniform distribution)	$-1,2\left[1+\frac{3\beta^2(\beta^2+6)}{(\beta^2+3)^2}\right]$
Trapezoidal with parameter α	$-1,2\,(1+\alpha^4)/(1+\alpha^2)^2$
Triangular	$-0,6$
Gaussian	0
Student's t with the number of degrees of freedom ν	$6/(\nu-4)$

4.3. Calculating the numerical value of the measurement result

The estimate of the measurand y in [1] is calculated by substituting the obtained estimates of the input values $x_1, x_2, ..., x_N$ in (1):

$$y = f(x_1, x_2, ..., x_N)\ . \tag{8}$$

With a nonlinear model, this estimation method gives an unbiased result only in the absence of uncertainty in the estimates of the input quantities. In the presence of significant uncertainties in the input quantities, this method leads to a bias in the estimate of the measurand [6]. This drawback can be eliminated by introducing a correction to the numerical value of the measurand. The procedure for calculating the correction, the criterion for its significance and the method for obtaining an unbiased estimate of the numerical value of the measurement result are given in Appendix B.

4.4. Calculation of the measurand standard uncertainty

The calculation of the measurand standard uncertainty is carried out in the GUM by the method of summing variances and covariances.

In the absence of correlation between the measurement results of the input quantities, the standard uncertainty of the measurand is found by the formula:

$$u(y) = \sqrt{\sum_{i=1}^{N} c_i^2 u_i^2}\ , \tag{9}$$

where $c_i = \partial y/\partial x_i$ are the sensitivity coefficients, which are the corresponding partial derivatives Y with respect to X_i, which are estimated at $X_1 = x_1, ..., X_N = x_N$.

If there is a correlation between the *k*-th and *l*-th input quantities, the standard uncertainty of the measurand $u(y)$ is found from the expression:

$$u(y) = \sqrt{\sum_{i=1}^{N} c_i^2 u_i^2 + 2c_l c_k \operatorname{cov}(x_k, x_l)}\ . \tag{10}$$

This method of estimation gives an unbiased result only with a linear model. With a nonlinear model in the presence of significant uncertainties in the input quantities, this method leads to a bias in the estimate of the measurand standard uncertainty [7], which can be eliminated by introducing a correction. The procedure for calculating the correction, the criterion for its significance and the procedure for obtaining an unbiased estimate of the standard uncertainty of the measurement result are given in Appendix C.

5. CALCULATION OF EXPANDED UNCERTAINTY

5.1. Kurtosis method

The expanded measurement uncertainty is calculated by this method using the formula:

$$U = k_p \cdot u(y), \tag{11}$$

where k_p is the coverage factor, which for a confidence level of 0,95 is calculated by formula [8]:

$$k_{0,95} = \begin{cases} 0,1085\eta^3 + 0,1\eta + 1,96, \text{ at } \eta < 0; \\ t_{0.95;(6/\eta+4)} \cdot \sqrt{\dfrac{3+\eta}{3+2\eta}}, \quad \text{at } \eta \geq 0, \end{cases} \tag{12}$$

and for a confidence level of 0,9545 – according to formula [8]:

$$k_{0,9545} = \begin{cases} 0,12\eta^3 + 0,1\eta + 2, \quad \text{at } \eta < 0; \\ t_{0,9455;(6/\eta+4)} \cdot \sqrt{\dfrac{3+\eta}{3+2\eta}}, \text{ at } \eta \geq 0, \end{cases} \tag{13}$$

It should be noted that for $\eta \geq 0$ with a deviation of no more than 2%, $k_{0,95} = 1,96$ and $k_{0,9545} = 2$ can be accepted.

In expressions (12-13) η is the kurtosis of the distribution of the measurand, determined in the absence of correlation between the results of measurements of the input quantities as:

$$\eta = \frac{\sum_{i=1}^{n} \eta_i c_i^4 u_i^4}{u^4(y)} . \tag{14}$$

and η_i is the kurtosis of the input quantity, taken from table 4.1; $u(y)$ is the standard uncertainty of the measurand, determined by formula (9).

If there is a correlation between the results of measurements of the *k*-th and *l*-th input quantities, the kurtosis of the distribution of the measurand should be calculated by formula [9]:

$$\eta = \frac{\sum\limits_{i=1;i\neq k;i\neq l}^{N} \eta_i c_i^4 u_i^4 + \eta_{kl}[c_k^2 u_k^2 + 2c_k c_l \operatorname{cov}(x_k, x_l) + c_l^2 u_l^2]^2}{u^4(y)} , \tag{15}$$

where $u(y)$ is the standard uncertainty of the measurand, determined by formula (10); $\eta_{kl} = \eta_k = \eta_l = 6/(n-5)$ is the kurtosis of the *k*-th (*l*-th) input quantity.

All information about the input and measured quantities obtained above are summarized in table 4, representing the uncertainty budget.

The deviation of the expanded uncertainty estimates obtained by the kurtosis method from the estimates obtained using the MCM does not exceed ± 2,5%.

It should be noted that the fulfillment of the inequality about the presence of a bias in the measurand (B7) indicates the asymmetry of its PDF. In this case, to find the expanded uncertainty, it is necessary to use the MMK.

As the kurtosis of the Student's PDF exists when the number of degrees of freedom is more than 4, then the kurtosis method is applicable for a number of repeated measurements of input quantities more than 5. With a smaller number of repeated measurements, the expanded uncertainty propagation law should be applied to calculate the expanded uncertainty [10] described in subsection 5.2.

Table 4

Measurement uncertainty budget for the kurtosis method

Input quantities	The values of input quantities	Standard uncertainties of input quantities	Kurtosis of input quantities	Sensitivity coefficients	Uncertainty contributions
X_1	x_1	$u(x_1)$	η_1	c_1	$c_1u(x_1)$
X_2	x_2	$u(x_2)$	η_2	c_2	$c_2u(x_2)$
…	…	…	…	…	…
X_N	x_N	$u(x_N)$	η_N	c_N	$c_Nu(x_N)$
Measurand	Measurand value	Standard uncertainty of the measurand	Measurand kurtosis	Coverage factor	Expanded uncertainty
Y	y	$u(y)$	η	k	U

5.2. The expanded uncertainty propagation law

Expanded Uncertainty Propagation Law (EUPL) is used to estimate the expanded uncertainty for a number of repeated observations of the input quantities $n \geq 4$.

The expression for calculating the expanded uncertainty for a probability of 0,95 in this case has the form of [10]:

$$U = \sqrt{U_A^2 + U_B^2}\,, \tag{16}$$

where $U_A(y)$, $U_B(y)$ are estimates of expanded uncertainties of type A and B of the measurand, respectively.

The expanded uncertainty of type A is calculated by the formula:

$$U_A = \sqrt{\sum_{i=1}^{N} \frac{t^2_{(0,95;\nu_i)} c_i^2 u^2(\varepsilon_i)(n_i - 3)}{n_i - 1}}\,, \tag{17}$$

where $u(\varepsilon_i)$ is determined by expression (4); $t^2_{(0,95;\nu_i)}$ is the Student's coefficient for the probability of 0,95 and the number of degrees of freedom $\nu_i = n_i - 1$.

The standard uncertainty of the correction for the random error of the measurand is determined by the formula:

$$u(\varepsilon) = \sqrt{\sum_{i=1}^{N} c_i^2 u^2(\varepsilon_i)}\,, \tag{18}$$

The expanded uncertainty of type B is calculated using the formula:

$$U_B = k_B \cdot u_B(y)\,, \tag{19}$$

where $u_B(y)$ is the type B standard uncertainty of the measurand:

$$u_B(y) = \sqrt{\sum_{i=1}^{N} c_i^2 u_B^2(x_i)}\,; \tag{20}$$

k_B is the type B coverage factor, calculated by the kurtosis method using the formula:

$$k_B = 0{,}1085\eta_B^3 + 0{,}1\eta_B + 1{,}96\,. \tag{21}$$

Here η_B is the kurtosis of the distribution of the composition of the components of the measurand, assessed by type B, equal to:

$$\eta_B = \frac{\sum_{i=1}^{N} \eta_B(x_i) c_i^4 u_B^4(x_i)}{u_B^4(y)} . \quad (22)$$

where $\eta_B(x_i)$ is the kurtosis of type B distribution of the *i*-th input quantity.

An estimate of the combined standard uncertainty of the measurand is found by the formula:

$$u(y) = \sqrt{[u(\varepsilon)]^2 + [u_B(y)]^2} . \quad (23)$$

When implementing EUPL, it is necessary to draw up two uncertainty budgets: for components estimated by type B (Table 5) and for corrections for random errors (Table 6).

Table 5

Measurement Uncertainty Budget for Type B Components

Input quantities	The values of input quantities	Standard uncertainties of input quantities	Kurtosis of input quantities	Sensitivity coefficients	Uncertainty contributions
X_1	x_1	$u_B(x_1)$	η_1	c_1	$c_1 u_B(x_1)$
X_2	x_2	$u_B(x_2)$	η_2	c_2	$c_2 u_B(x_2)$
…	…	…	…	…	…
X_N	x_N	$u_B(x_N)$	η_N	c_N	$c_N u_B(x_N)$
Measurand	Measurand value	Type B standard uncertainty of the measurand	Type B measurand kurtosis	Type B coverage factor	Type B expanded uncertainty
Y	y	$u_B(y)$	η_B	k_B	U_B

Table 6

Measurement Uncertainty Budget for Random Errors

Input quantities	The values of input quantities	Standard uncertainty corrections to input quantities	Number of degrees of freedom of input quantities	Sensitivity coefficients	Uncertainty contributions of correction
ε_1	0	$u(\varepsilon_1)$	ν_1	c_1	$c_1 u(\varepsilon_1)$
ε_2	0	$u(\varepsilon_2)$	ν_2	c_2	$c_2 u(\varepsilon_2)$
…	…	…	…	…	…
ε_N	0	$u(\varepsilon_N)$	ν_N	c_N	$c_N u(\varepsilon_N)$
Measurand	Measurand value	Standard uncertainty of the correction to the measurand			Type A expanded uncertainty
ε	0	$u(\varepsilon)$			U_A

If there is a correlation between the measurements of two input quantities X_k, X_l, then the standard uncertainty for the random error of the measurand is determined by formula [10]:

$$u(\varepsilon) = \sqrt{\sum_{i=1}^{N} c_i^2 u^2(\varepsilon_i) + 2c_k c_l \operatorname{cov}(x_k, x_l)} \;, \tag{24}$$

and the expanded uncertainty of type A is calculated as:

$$U_A = \sqrt{\sum_{i=1}^{N} \frac{t_{(0,95;\nu_i)}^2 c_i^2 u^2(\varepsilon_i)(n_i - 3)}{n_i - 1} + 2t_{(0,95;\nu)}^2 c_k c_l \operatorname{cov}(x_k, x_l) \frac{(n-3)}{(n-1)}} \;, \tag{25}$$

where $n_k = n_l = n$.

The deviation of the expanded uncertainty estimates obtained by this method from the estimates obtained using the MCM does not exceed ± 4.5%.

A bibliography with examples of the estimation of measurement uncertainty by both methods is given in Appendix **E**.

Finding the parameters of distributions by coverage factor

The parameter $\alpha = u_2/u_1$ of the trapezoidal PDF with standard uncertainty u_{trap}, where $u_1 = u_{trap}\Big/\sqrt{1+\alpha^2}$, $u_2 = \alpha u_1$ are the standard uncertainties of the two uniform PDFs that make up the trapezoidal PDF, can be found from Fig. A1.

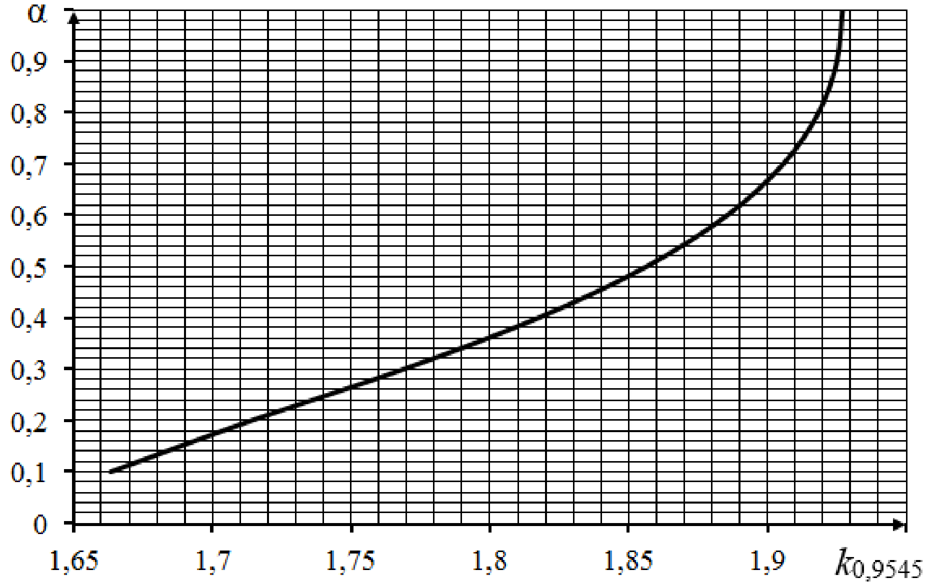

Figure A1 – Nomogram for finding the parameter $\alpha = u_2/u_1$ of the trapezoidal PDF, by the value of $k_{0,9545}$

The number of degrees of freedom ν of the Student's PDF for a probability of 0,9545 can be found from Fig. A2.

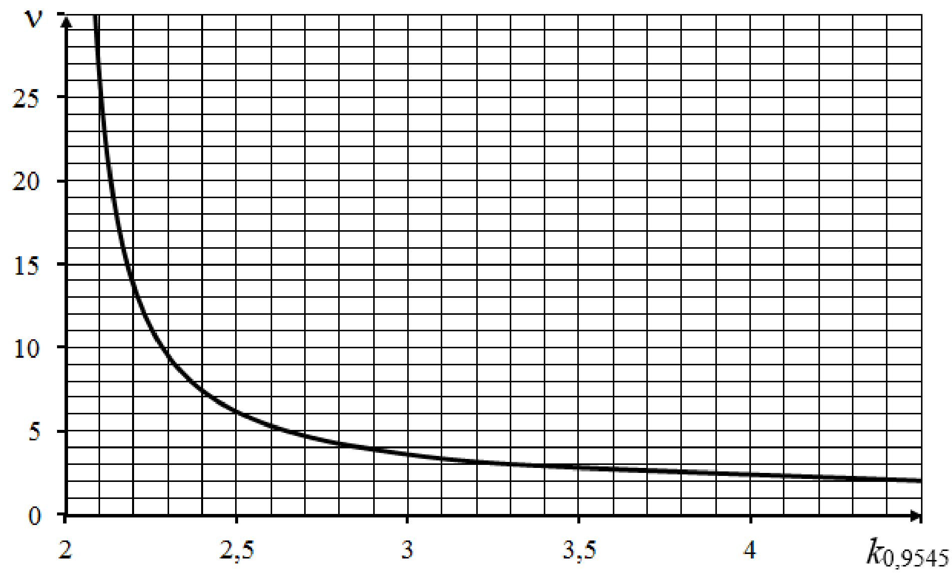

Figure A2 - Nomogram for finding the effective number of degrees of freedom ν of the Student's PDF by the coverage coefficient $k_{0,9545}$

Taking into account the bias of the numerical value of the measurand with a nonlinear model equation

The expression for the value of the bias of the estimate of the measurand in the absence of correlation between the input values has the form of [6]:

$$\Delta_y = -\frac{1}{2}\sum_{i=1}^{N} c_{ii}u_i^2 \ , \qquad \text{(B1)}$$

where $c_{ii} = \partial^2 y / \partial x_i^2$ is the second-order partial derivative Y with respect to X_i which is estimated at $X_1 = x_1, \ldots, X_N = x_N$; u_i is standard uncertainty of X_i.

The bias value can be calculated by the method of partial increments [6] according to the formula:

$$\Delta_y = \sum_{j=1}^{N}\left\{\frac{f[x_1,\ldots,(x_j+u_j),\ldots,x_N]+f[x_1,\ldots,(x_j-u_j),\ldots,x_N]}{2} - f[x_1,x_2,\ldots,x_N]\right\}. \qquad \text{(B2)}$$

If there is a correlation between the *k*-th and *l*-th input quantities, the expression for the bias value of the estimate of the measurand has the form

$$\Delta(y) = -\left[\frac{1}{2}\sum_{i=1}^{N} c_{jj}u_j^2 + c_{kl}r_{kl}u_k u_l\right], \qquad \text{(B3)}$$

where $c_{kl} = \partial^2 y / (\partial x_k \partial x_l)$ is the second-order mixed partial derivative Y with respect to X_k, X_l, which is estimated at $X_1 = x_1, \ldots, X_N = x_N$; r_{kl} is the correlation coefficient between the measurement results X_k, X_l, calculated by the formula:

$$r_{kl} = \frac{\sum_{r=1}^{n}(x_{kr}-\overline{x}_k)(x_{lr}-\overline{x}_l)}{\sqrt{\sum_{r=1}^{n}(x_{kr}-\overline{x}_k)^2 \sum_{r=1}^{n}(x_{lr}-\overline{x}_l)^2}}; \qquad \text{(B4)}$$

u_k, u_l are standard uncertainties caused by the observed dispersion of readings of the *k*-th and *l*-th input quantities, and:

$$u_k = \sqrt{\frac{n-1}{n-3}}\frac{s_k}{\sqrt{n}}, \qquad \text{(B5)}$$

$$u_l = \sqrt{\frac{n-1}{n-3}}\frac{s_l}{\sqrt{n}}. \qquad \text{(B6)}$$

The resulting bias value $\Delta(y)$ is compared to the unbiased standard uncertainty estimate $u_0(y)$, which is obtained in the next section. If the inequality holds:

$$|\Delta(y)| \geq \frac{1}{3}u_0(y), \qquad \text{(B7)}$$

it is necessary to take into account the bias $\Delta(y)$ as a correction to y (8), obtaining an unbiased estimate of the measurand using the formula:

$$y_0 = y - \Delta(y). \qquad \text{(B8)}$$

Accounting for the bias of the standard uncertainty of the measurand at a nonlinear model equation

The bias of the estimate of the standard uncertainty of the measurand is calculated by formula [7]:

$$\Delta(u^2) = \left[\frac{1}{4} \sum_{i=1}^{N} c_{ii}^2 (\eta_i + 2) u_i^4 + \sum_{i=2}^{N} \sum_{j=1}^{i} c_{ij}^2 u_i^2 u_j^2 \right], \quad \text{(C1)}$$

where η_i is the kurtosis of the PDF of the *i*-th input quantity, which is taken from table 3.

The resulting bias value is compared with the value $u^2(y)$. If the inequality holds:

$$\left| \Delta(u^2) \right| \geq \frac{1}{9} u^2(y), \quad \text{(C2)}$$

the bias $\Delta(u^2)$ must be taken into account as a correction to $u^2(y)$, obtaining an unbiased estimate of the standard uncertainty of the measurand using the formula:

$$u_0^2(y) = u^2(y) + \Delta(u^2). \quad \text{(C3)}$$

To facilitate calculations by formulas (C1)–(C3), it is advisable to use the partial increments method.

In this case, the first-order differential partial derivative of the measurand with respect to the *j*-th input quantity will be equal to:

$$c_j^* = \frac{f[x_1, ..., (x_j + u_j), ..., x_N] - f[x_1, ..., (x_j - u_j), ..., x_N]}{2u_j}. \quad \text{(C4)}$$

The second-order differential partial derivative of the measurand with respect to the *j*-th input quantity will be equal to:

$$c_{jj}^* = \frac{1}{u_j^2} \{ f[x_1, ..., (x_j + u_j), ..., x_N] - 2f(x_1, x_2, ..., x_N) + f[x_1, ..., (x_j - u_j), ..., x_N] \}. \quad \text{(C5)}$$

The difference mixed partial derivative of the second order of the measurand with respect to the *j*-th and *i*-th input quantities will be equal to:

$$c_{ji}^* = \frac{1}{4u_j u_i} \{ f[x_1, ..., (x_j + u_j), ..., (x_i + u_i), ..., x_N] - f[x_1, ..., (x_j + u_j), ..., (x_i - u_i), ..., x_N] -$$

$$- f[x_1, ..., (x_j - u_j), ..., (x_i + u_i), ..., x_N] + f[x_1, ..., (x_j - u_j), ..., (x_i - u_i), ..., x_N] \}. \quad \text{(C6)}$$

APPENDIX D

Student's coefficients for the number of degrees of freedom v and probabilities 0,95 and 0,9545

v	0,95	0,9545	v	0,95	0,9545	v	0,95	0,9545	v	0,95	0,9545
1	12,71	13,97	**11**	2,20	2,25	**21**	2,08	2,13	**35**	2,03	2,07
2	4,30	4,53	**12**	2,18	2,23	**22**	2,07	2,12	**40**	2,02	2,06
3	3,18	3,31	**13**	2,16	2,21	**23**	2,07	2,11	**45**	2,01	2,06
4	2,78	2,87	**14**	2,14	2,20	**24**	2,06	2,11	**50**	2,01	2,05
5	2,57	2,65	**15**	2,13	2,18	**25**	2,06	2,11	**60**	2,00	2,04
6	2,45	2,52	**16**	2,12	2,17	**26**	2,06	2,10	**70**	1,99	2,04
7	2,36	2,43	**17**	2,11	2,16	**27**	2,05	2,10	**80**	1,99	2,03
8	2,31	2,37	**18**	2,10	2,15	**28**	2,05	2,09	**90**	1,99	2,03
9	2,26	2,32	**19**	2,09	2,14	**29**	2,05	2,09	**100**	1,98	2,03
10	2,23	2,28	**20**	2,09	2,13	**30**	2,04	2,09	∞	1,96	2,00

APPENDIX E

List of publications with examples of estimation of measurement uncertainty by the proposed methods

E1. Zakharov I., Botsiura O., Brikman A., Zakharov O. Evaluation of expanded uncertainty at glass thermometer calibration // Ukrainian Metrological Journal, 2019, No 4, 23-28. DOI: 10.24027/2306-7039.4.2019.195953.

E2. Zakharov I., Botsiura O., Semenikhin V. Measurement uncertainty evaluation by kurtosis method at calibration of electrical resistance standards using a comparator// Ukrainian Metrological Journal, 2020, No 1, 12-16. DOI: 10.24027/2306-7039.1.2020.204166.

E3. Botsiura O.A., Zakharov I.P. Increasing the Reliability of Evaluation of Expanded Uncertainty in Calibration of Measuring Instruments //Measurement Techniques, 2020 Volume: 63, Issue: 6, pp. 414-420. DOI 10.1007/s11018-020-01803-2.

E4. Zakharov I.P., Botsiura O.A., Patsenko O.M. Measurement uncertainty evaluation at mass calibration // Ukrainian Metrological Journal, 2020, No 3, 36-41. DOI: 10.24027/2306-7039.3.2020.216839.

E5. Zakharov I.P., Botsiura O.A., Tsybina I.Yu., Zakharov O.O. Measurement uncertainty evaluation at micrometer calibration // Ukrainian Metrological Journal, 2020, No 3a, c. 196-201. DOI: 10.24027/2306-7039.3A.2020.220313.

E6. Zakharov I., Botsiura O., Semenikhin V., Fomenko V. Considering of the input quantities distributions in the procedure for measurements uncertainty evaluating on the example of resistance box calibration // Ukrainian Metrological Journal, 2020, No 4, c. 3-8. DOI: 10.24027/2306-7039.4.2020.224189.

E7. Zakharov I., Neyezhmakov P., Botsiura O. Expanded Uncertainty Evaluation Taking into Account the Correlation Between Estimates of Input Quantities // Ukrainian Metrological Journal, 2021, No 1, 4-8. DOI: https://doi.org/10.24027/2306-7039.1.2021.228134.

E8. Zakharov I., Botsiura O., Neyezhmakov P., Obtaining Uncertainty Estimates Compatible with Estimates of Monte Carlo Method // Measurement-2019: Proceedings of the 12th International Conference, Smolenice, Slovakia, 27-29 May 2019, p. 47-50.

E9. Zakharov I.P., Neyezhmakov P.I., Botsiura O.A. Revision of GUM: the suggested algorithm for processing measurement results // 2019 IEEE 8th International Conference on Advanced Optoelectronics and Lasers (CAOL), 6-8 Sept. 2019, Sozopol, Bulgaria, pp. 632 – 635. DOI: 10.1109/CAOL46282.2019.9019421. Electronic ISSN: 2160-1534. Print on Demand(PoD) ISSN: 2160-1518.

E10. Zakharov I., Botsiura O., Zadorozhna I. Measurement Uncertainty Evaluation at Gauge Block Calibration // 2019 XXIX International Scientific Symposium "Metrology and Metrology Assurance" (MMA), 6-10 Sept. 2019, Sozopol, Bulgaria, pp. 19-22. DOI: 10.1109/MMA.2019.8936023.

E11. Zakharov I., Neyezhmakov P., Botsiura O. Expanded Uncertainty Evaluation Taking into Account the Correlation Between Estimates of Input Quantities // Sensor and Measurement Science International Conference (SMSI 2020), 22-25 June 2020, Nuremberg, Germany, pp. 351-352. DOI 10.5162/SMSI2020/P4.7.

E12. Zakharov I., Serhiienko M., Chunikhina T. Measurement uncertainty evaluation by kurtosis method at calibration of a household water meter // Metrology and Metrology Assurance (MMA-2020): Proceedings of 2020 XXX International Scientific Symposium, Sozopol, Bulgaria, 7-11 Sept. 2020, pp. 83-86. DOI: 10.1109/MMA49863.2020.9254260.

E13. Zakharov I., Botsiura O., Semenikhin V. Method of kurtosis in estimating the measurement uncertainty during evaluation of the electrical resistance measures using a potentiometer // Ukrainian Metrological Journal, 2021, No 2, pp. 30-34. DOI: 10.24027/2306-7039.2.2021.236078.

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10. Zakharov I., Botsyura O. Estimation of expanded uncertainty in measurement when implementing a Bayesian approach // Measurement Techniques, 2018, Volume: 61, Issue: 4, pp. 342-346.

Information

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